Higher Approximations in the Asymptotic Theory of Propeller Noise

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This paper extends earlier work on the asymptotic theory of the acoustic radiation from a single-rotation propeller (in the many-blade limit $B \to \infty$) in two ways. First, a correction is found to the dominant Mach radius field in supersonic operating conditions; the correction is associated with the blade tip and its inclusion brings the asymptotic prediction into fine agreement with numerical evaluation of the radiation integrals, in addition to giving physical insight and the possibility of control, at negligible computing expense. Second, an expression is derived that provides a transition across the Mach condition (from subsonic to supersonic radiation characteristics), and from this, composite expressions are given, valid across the entire range of operating conditions.

Nomenclature

Ai = Airy function
B = number of blades
J_{mB} = Bessel function of first kind and order mB
M_{rt} = tip relative Mach number
M_t = tip rotational Mach number
M_x = flight (axial) Mach number
m = harmonic index
P_m = normalized complex amplitude of far-field pressure in mth harmonic
$S(z)$ = normalized radial source function
z = normalized radial (spanwise) coordinate
z_0 = normalized hub radius
z^* = Mach radius, defined by Eq. (2)
β_t = $\text{sech}^{-1}(1/z^*)$, see Eq. (16)
Δ = coordinate defined by Eq. (18)
θ = angular location of observer at emission time
λ_t = $\sec^{-1}(1/z^*)$, see Eq. (14)
μ = tip-source exponent

I. Introduction

IN this paper we address the problems of determining higher approximations for the noise of a supersonic propeller and of determining approximations that change continuously, and preferably smoothly, between subsonic and supersonic operating conditions. The underlying approximation scheme is that of large blade number, exploited in earlier papers[1-3] for subsonic and supersonic operating conditions. Dealing with the first problem, we place the intuitive asymptotic arguments of Ref. 3 on a firmer footing, in a formal sense at least, obtaining estimates of the orders of the errors incurred. We shall also derive an explicit expression for the next term (after the dominant Mach-radius term), showing that it corresponds to radiation from the tip of a supersonic propeller and that its inclusion brings our asymptotic predictions into fine agreement with purely numerical calculations. The need to deal with the second problem is illustrated in Fig. 1, where we show a plot of propeller noise against tip rotational relative Mach number as calculated using the asymptotic approximations of Refs. 1 and 3. Here we switch from the subsonic to the supersonic approximation at the Mach condition, and at this point the curve is discontinuous—indeed, the subsonic approximation of Ref. 1 is singular here. The smoothing of this singular jump is discussed in Sec. III.

II. Higher Asymptotic Approximations

Bessel Function Approximations

The analysis refers to a regular B-bladed propeller, with tip rotational Mach number M_t and axial (flight) Mach number M_x. A frequency-domain approach is adopted, for which Hanson[4] has given convenient forms for the radiation integrals from blade surface monopole and dipole sources corresponding to blade thickness and loading distributions. These inte-

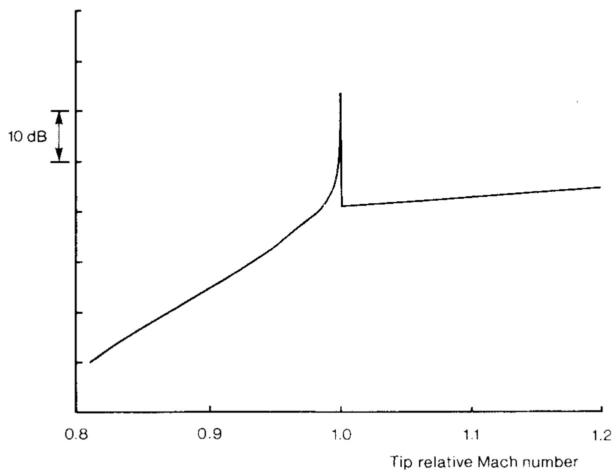

Fig. 1 Asymptotic calculation of propeller noise vs tip relative Mach number (radiation angle θ = 90 deg). For $M_{rt} < 1$ we use Eq. (16), and for $M_{rt} \geq 1$ we use Eq. (17).

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grals can be taken in the form

$$P_m = \int_{z_0}^{1} S(z)\, J_{mB}\left(mB\,\frac{z}{z^*}\right) \mathrm{d}z \tag{1}$$

where P_m refers to the complex amplitude of the mth harmonic of blade passing frequency, the integral is taken along the span of a single blade from $z = z_0$ (the hub) to $z = 1$ (the tip), $S(z)$ denotes a typical spanwise distribution of loading or thickness (see Refs. 1 and 3), and z^* is the Mach radius for radiation in direction θ from the flight direction defined by

$$z^* = \frac{1 - M_x \cos\theta}{M_t \sin\theta} \tag{2}$$

If $z^* > 1$, the Bessel function in Eq. (2) decreases exponentially inboard from the tip, and as $B \to \infty$ the radiation is entirely controlled by the tip behavior, decreasing exponentially with increase of mB. These are the characteristics of *subsonic* radiation, and explicit formulas for this case were given in Ref. 1 and favorably compared both with experiment and full numerical evaluation of Eq. (1). If $z^* < 1$, it was argued in Ref. 3 that, since the Bessel function is exponentially small inboard of $z = z^*$ and oscillates very rapidly outboard of $z = z^*$, the dominant radiation must come from the immediate vicinity of $z = z^*$ (referred to as the Mach radius, a station that, at some time in each revolution of the propeller, approaches the observer at precisely sonic speed), and thus that

$$P_m \sim S(z^*) \int_0^{\infty} J_{mB}\left(mB\,\frac{z}{z^*}\right) \mathrm{d}z = \frac{z^* S(z^*)}{|mB|} \tag{3}$$

a prediction that was also quite favorably compared with full numerical evaluation in Ref. 3. However, that comparison revealed oscillations in levels not predicted by Eq. (3). We aim here to show that these oscillations can be accurately predicted by a higher-order asymptotic approximation and thus to justify further the large-B limit as a useful method for analyzing propeller noise (and also, of course, to obtain a highly accurate noise prediction).

It should be pointed out that all "spanwise interference effects" will be excluded from consideration in this paper. Such effects are introduced if the blades are swept or if the blades do not have compact chord relative to the relevant acoustic wavelength. A first estimate of the modification to Eq. (3) that is associated with chordwise noncompactness was given in Ref. 3; further study of spanwise interference effects is deferred to a separate paper.

Since the Bessel function in Eq. (1) tends, in general, to control the radial integrations, we consider asymptotic approximations for $J_\nu(\nu\sigma)$ that cover all values of $\sigma = z/z^*$ as $\nu \to \infty$. We need consider only positive ν, since only integer values $\nu = mB$ are of interest to us, and for these $J_{-\nu}(-\nu\sigma) = J_\nu(\nu\sigma)$.

First suppose that $\sigma < 1$ and $\sigma = O(1)$. We write $\sigma = \operatorname{sech}\beta$ and use the Debye asymptotic approximation. The leading term is[5]

$$J_\nu(\nu \operatorname{sech}\beta) \sim \frac{\exp[\nu(\tanh\beta - \beta)]}{(2\pi\nu \tanh\beta)^{1/2}} \tag{4}$$

and inspection of higher terms indicates that this approximation is nonuniform when $\beta = O(\nu^{-1/3})$. Next suppose that $\beta = O(\nu^{-1/3})$ and put $\beta = \kappa\nu^{-1/3}$ where $\kappa = O(1)$. Since β is small, we obtain

$$J_\nu(\nu \operatorname{sech}\beta) \sim J_\nu\left(\nu - \frac{\kappa^2}{2}\nu^{1/3}\right) + O(\nu^{-1}) \text{ as } \nu \to \infty \tag{5}$$

where the next term comes from a Taylor expansion and Sec. 9.3.27 of Ref. 5. From section 9.3.23,[5] we find that Eq. (5) reduces further to

$$J_\nu(\nu \operatorname{sech}\beta) \sim \frac{2^{1/3}}{\nu^{1/3}} Ai\left(\frac{\kappa^2}{2^{2/3}}\right) + O(\nu^{-1}) \tag{6}$$

or equivalently,

$$J_\nu(\nu\sigma) \sim \frac{2^{1/3}}{\nu^{1/3}} Ai[2^{1/3}\nu^{2/3}(1 - \sigma)] \tag{7}$$

It can easily be shown that the two equations, Eqs. (4) and (7), match to leading order in the overlap range $\nu^{-1/3} << \beta << 1$. It would be possible to use this fact to form a *composite* smooth approximation that reduces to Eq. (4) or Eq. (7) as appropriate. It is preferable, however, simply to switch between them at any point $\sigma = \sigma_-$ in the overlap range. The value of σ_- is otherwise arbitrary and will disappear when the integrals involving Eqs. (4) and (7) are calculated and added, leaving a smoothly varying asymptotic expression for the pressure.

We assume that the value $\sigma = \sigma_0$ corresponding to the hub $z = z_0$ does not lie in the overlap range, i.e., that the hub and Mach radius are not close; this condition is well satisfied unless the tip rotational Mach number is very high indeed.

We now turn to the case where $\sigma > 1$ and $\sigma = O(1)$ and define λ by $\sigma = \sec\lambda$. The asymptotic expansion for $J_\nu(\nu\sigma)$ is then given in Sec. 9.3.15 of Ref. 5 and is again nonuniform [when $\lambda = O(\nu^{-1/3})$]. However, for $\nu^{-1/3} << \lambda$ we can use the leading approximation

$$J_\nu(\nu \sec\lambda) \sim \left(\frac{2}{\pi\nu \tan\lambda}\right)^{1/2} \cos[\nu(\tan\lambda - \lambda) - \pi/4] \tag{8}$$

For $\lambda = O(\nu^{-1/3})$ analysis shows that then we can again use Eq. (7). The two equations, Eqs. (7) and (8), match in the range $\nu^{-1/3} << \lambda << \pi/2$, and we switch between them at any point in this range, the corresponding value of σ being denoted by σ_+. We suppose here that the value $\sigma_t = 1/z^*$ (corresponding to the tip) does not lie in the overlap range; see Sec. III for the case when it does.

The radial range of integration has correspondingly now been split into three distinct regions: a region defined by $\sigma < \sigma_-$, inboard of the Mach radius, in which the Mach number component $M_x \cos\theta + zM_t \sin\theta$ toward the observer is always less than unity and where the Bessel function can be approximated by Eq. (4); a Mach-radius region, defined by $\sigma_- < \sigma < \sigma_+$, in which that Mach number component is always close to unity and where the Bessel function can be approximated by Eq. (7); and a region defined by $\sigma > \sigma_+$, outboard of the Mach radius, in which the Mach number component is always greater than unity and where the Bessel function can be approximated by Eq. (8). Observe that the boundaries between these regions change with θ.

The acoustic radiation [with harmonic components $P_m^{(1)}$] from the inboard region is very similar to that discussed in Ref. 1 (with appropriate changes from parameter values at the tip to values at z_- defining the upper bound of the inboard region) and will not, therefore, be further considered in any detail. If no Mach radius exists for a given θ, then the radiation in direction θ is entirely given by the $P_m^{(1)}$ and is tip dominated. If a Mach radius does exist, then, as we shall shortly see, the contribution from $z = z_-$ is actually canceled, at least to the first two nontrivial orders, by a contribution from the Mach-radius region, so that there is actually no significant radiation associated with the region of the blade inboard of the Mach radius.

Radiation from the Mach Radius

Here, for reasons explained in Ref. 3, all spanwise interference effects will be ignored. Thus either the blades are assumed to be chordwise compact, or the special case k_x = const is taken.[3] In either case the radiation is given by Eq. (1), with an appropriate definition of $S(z)$, this containing no exponentially rapid variation with mB. Expanding $S(z)$ about the Mach radius, the harmonic components $P_m^{(2)}$ from the region around the Mach radius are given by

$$P_m^{(2)} \sim S(z^*) \int_{z_-}^{z_+} J_{mB}\left(mB\frac{z}{z^*}\right) dz + S'(z^*) \int_{z_-}^{z_+} (z - z^*) J_{mB}\left(mB\frac{z}{z^*}\right) dz + \ldots\ldots \qquad (9)$$

where $z_\pm = z^* \sigma_\pm$.

Now we write $z/z^* = 1 + x/2^{1/3}(mB)^{2/3}$, so that

$$J_{mB}\left(mB\frac{z}{z^*}\right) = J_{mB}\left[mB + \left(\frac{x}{2^{1/3}}\right)(mB)^{1/3}\right] = \frac{2^{1/3}}{(mB)^{1/3}} Ai(-x)\left\{1 - \frac{1}{5}\left(\frac{x}{2^{1/3}}\right) + O\left[\frac{x^5}{(mB)^{4/3}}\right]\right\} + \frac{2^{2/3}}{mB} Ai'(-x)\left\{\frac{3}{10}\left(\frac{x}{2^{1/3}}\right)^2 + O\left[\frac{x^5}{(mB)^{2/3}}\right]\right\} \qquad (10)$$

which we introduce into Eq. (9). Our aim now is simply to evaluate the resulting expression keeping all terms of order $(mB)^{-3/2}$ and larger, as $mB \to \infty$. The reason is that we shall find that the tip contribution is of order $(mB)^{-3/2}$, and we wish to keep this and all larger contributions.

The asymptotic evaluation of the integral in Eq. (9), with Eq. (10), is complicated and will be simply summarized here. First, note that if we are to have overlap to leading order between Eqs. (7) and (8), then the value of x corresponding to z_+ must satisfy $1 \ll x_+ \ll (mB)^{2/3}$. But if we are to keep all terms explicitly quoted in Eq. (10) and neglect the terms implied by the order symbol, then this overlap range shrinks drastically to second order and becomes $1 \ll x_+ \ll (mB)^{1/6}$, and there is a corresponding restriction on x_-. When $x_\pm$ are restricted in this way, it is straightforward to use standard techniques, including the term-by-term integration of the asymptotic expansion of $Ai(-x)$, to find the required asymptotic expansion of the $P_m^{(2)}$. To evaluate the integral of $xAi(-x)$, one introduces an integral representation for $Ai(-x)$ and performs the x integration to get

$$\int_{x_-}^{x_+} xAi(-x)\, dx = -\frac{x_+}{\pi}\int_0^\infty \frac{1}{t}\left[\sin\left(\frac{t^3}{3} - x_+ t\right) + \sin\left(\frac{t^3}{3} + x_+ t\right)\right] dt + \frac{1}{\pi}\int_0^\infty \frac{1}{t^2}\left[\cos\left(\frac{t^3}{3} - x_+ t\right) - \cos\left(\frac{t^3}{3} + x_+ t\right)\right] dt \qquad (11)$$

plus a corresponding set of terms involving x_-. Now evaluate Eq. (11) by stationary phase [or integration by parts for the terms with argument $(t^3/3 + x_+ t)$]; a little care is needed as the third and fourth integrals do not exist separately, and the range of integration must be appropriately divided, but the upshot is that the stationary phase point $t = x_+^{1/2}$ provides the dominant contribution,

$$\int_{x_-}^{x_-} xAi(-x)\, dx = O(x_+^{1/4})$$

When all this is carried out we find, for appropriately restricted $x_\pm$, that

$$P_m^{(2)} = \frac{z^* S(z^*)}{|mB|} + F_+(x_+, mB) + F_-(x_-, mB) + o(mB)^{-3/2}$$

where $F_\pm$ are larger than $(mB)^{-3/2}$ but smaller than $(mB)^{-1}$, as $mB \to \infty$ for the restricted $x_\pm$. However, the terms in $F_-(x_-, mB)$ are precisely canceled, with an error smaller than $(mB)^{-3/2}$, when the "subsonic" field $P_m^{(1)}$ from the integral over (z_0, z_-) is added. Thus,

$$P_m^{(1)} + P_m^{(2)} = \frac{z^* S(z^*)}{|mB|} + F_+(x_+, mB) + o(mB)^{-3/2} \qquad (12)$$

and there is no identifiable radiation, at least through order $(mB)^{-3/2}$, from the inboard part of the blade. Since the radiation $P_m^{(1)}$ analyzed in Ref. 1 is exponentially small in mB, a similar cancellation occurs to all algebraic orders in $(mB)^{-1}$, and no significant radiation is felt in direction θ from any section of the blade inboard of the Mach radius for angle θ.

Radiation from the Outboard Section

Here the harmonic components are given by

$$P_m^{(3)} = \int_{z_+}^{1} S(z) J_{mB}\left(mB\frac{z}{z^*}\right) dz$$

where, if $z = z^* \sec\lambda$,

$$J_{mB}\left(mB\frac{z}{z^*}\right) \sim \left(\frac{2}{\pi mB \tan\lambda}\right)^{1/2} \times [L(mB,\lambda)\cos\Psi + M(mB,\lambda)\sin\Psi]$$

$$\Psi = mB(\tan\lambda - \lambda) - \frac{\pi}{4}$$

$$L(\nu,\lambda) = 1 + O\left(\frac{\cot^6\lambda}{\nu^2}\right)$$

$$M(\nu,\lambda) = \frac{5\cot^3\lambda + 3\cot\lambda}{24\nu} + O\left(\frac{\cot^9\lambda}{\nu^3}\right)$$

Since there is no point on $[z_+, 1]$ at which the phase Ψ is stationary, we integrate repeatedly by parts, picking up contributions from the lower limit z_+, and from the blade tip, at which $\lambda = \lambda_t$. This will produce a formal asymptotic expansion provided $S(z)$ has a Taylor series expansion in integral powers of $(1 - z)$ around the tip. A modification is needed in other cases and will be given in a moment.

The leading contribution from the tip is $O(mB)^{-3/2}$. Contributions from z_+ include several terms larger than this, but when all are converted to the variable x_+ and expanded for $mB \to \infty$ with x_+ appropriately restricted as before, we find

$$P_m^{(3)} = -F_+(x_+, mB) + [O(mB)^{-3/2} \text{ tip-contribution}] + o(mB)^{-3/2} \qquad (13)$$

Thus the contribution from the outboard blade section is associated with the tip only, at least to $O(mB)^{-3/2}$.

The final form for the radiation from the whole blade takes the form

$$P_m = P_m^{(1)} + P_m^{(2)} + P_m^{(3)} = \frac{z^* S(z^*)}{|mB|} + \left(\frac{2}{\pi mB \tan\lambda_t}\right)^{1/2} \frac{z^* S(1)}{mB \sin\lambda_t} \times \sin\left[mB(\tan\lambda_t - \lambda_t) - \frac{\pi}{4}\right] + o(mB)^{-3/2} \qquad (14)$$

where the first term comes from the Mach radius, the second from the tip, with $\lambda_t = \sec^{-1}(1/z^*)$. The asymptotic prediction afforded by Eq. (14) agrees remarkably well with full nu-

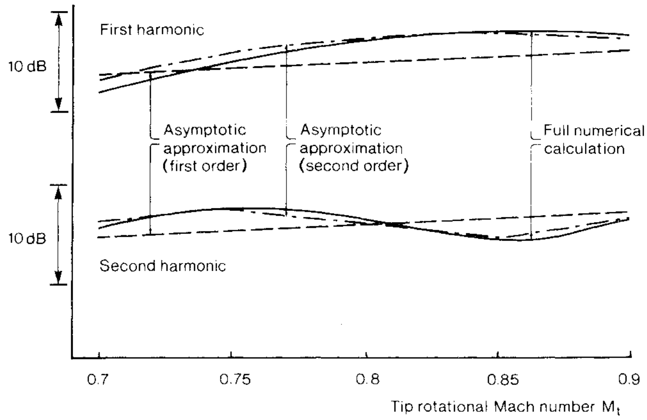

Fig. 2 Comparison between first-order [Eq. (3)] and second-order [Eq. (14)] asymptotic predictions and numerical calculations for the noise of a supersonic propeller with 12 straight blades; flight Mach number $M_x = 0.8$, radiation angle $\theta = \cos^{-1} M_x$.

merical calculations, obtained using standard IMSL or NAG routines. This is shown clearly in Fig. 2.

If the tip source has the form $S(z) \sim \bar{S}(1 - z)^\mu$ (where μ may be nonintegral), standard theory[6] indicates that the tip radiation is given by approximating the Bessel function as done earlier in this section, linearizing its phase $[mB(\tan\lambda - \lambda) - (\pi/4)]$ about its tip value, replacing $S(z)$ by its tip variation and integrating over z from $-\infty$ to 1. The result is

$$P_m^{\text{tip}} \sim \left(\frac{2}{\pi mB \tan\lambda_t}\right)^{1/2} \left|\frac{z^*}{mB \sin\lambda_t}\right|^{\mu+1} \times \bar{S} \sin\left[mB(\tan\lambda_t - \lambda_t) - \frac{\pi}{4}\right] \tag{15}$$

and checks with Eq. (14) when $\mu = 0$.

It is extremely tedious to carry these calculations further, and we have not been able to determine the actual magnitude of the next term beyond those quoted in Eq. (14). Certainly, there are higher-order terms from the tip, and these are easily calculated, though probably not needed. However, there are also likely to be corrections at higher order to the radiation from the Mach-radius region. These would arise from an incomplete cancellation between terms like $F_+(x_+, mB)$ arising from $P_m^{(2)}$ and $P_m^{(3)}$, such lack of cancellation being possible if the remainder does not involve the (arbitrary) x_+. Figure 2 indicates, however, that the essential features have been captured by the first two terms of the expansion, quoted explicitly in Eqs. (14) or (15) and associated directly with the Mach radius and with the tip.

Two further points must be made here. First, Fig. 2 indicates that, at fixed M_x and θ, the harmonic amplitudes P_m increase with tip rotational Mach number M_t. It was shown in Ref. 3, however, that chordwise noncompactness effects modify this conclusion, at least as far as the Mach radius radiation is concerned, and cause the P_m to decrease (slowly) with M_t. This would apply particularly to the high harmonics. Consequently, Fig. 2 must be seen as a comparison between asymptotics and numerics in circumstances where chordwise noncompactness effects are negligible. Second, while one can generally expect the kind of accuracy shown in Fig. 2 for values of mB equal to 10 or more, one could not expect reasonable accuracy for the lowest harmonics of a general aviation propeller with $B = 2$. For such cases, one would have to evaluate the radiation integrals numerically, a trivial task for small mB.

III. Transition Through the Mach Condition

For any values of the Mach numbers M_x and M_t and of the radiation angle θ, one may define a Mach radius z^* by Eq. (2). If $z^* > 1$ and is not close to 1, then the radiation is tip dominated and given by the subsonic-propeller theory of Ref. 1, with

$$P_m \sim \frac{\bar{S} \exp[mB(\tanh\beta_t - \beta_t)]}{(2\pi mB \tanh\beta_t)^{1/2}} \frac{\mu!}{(mB \tanh\beta_t)^{\mu+1}} \tag{16}$$

Here $z = z^* \operatorname{sech}\beta$ and the tip $z = 1$ corresponds to $\beta = \beta_t$, while $\bar{S}$ and μ are defined as before by an assumed tip distribution, $S(z) \sim \bar{S}(1 - z)^\mu$ as $z \to 1-$. Now if the tip total Mach number $(M_x^2 + M_t^2)^{1/2} > 1$, then z^* may decrease to values less than 1 as these parameters change through values corresponding to the Mach condition $z^* = 1$, and when $z^* < 1$ and is not close to 1, the dominant term for the radiation is that from the Mach radius,

$$P_m \sim \frac{z^* S(z^*)}{(mB)} \tag{17}$$

(where as previously mB must be read as $|mB|$ for $m < 0$). Here we derive a formula that takes us smoothly from Eq. (16) to Eq. (17) as z^* decreases through 1.

The essential point is that the scale of the Mach-radius region is $O(mB)^{-2/3}$ [cf. Eq. (7)]. Therefore the transition between Eqs. (16) and (17) must be described in terms not of fixed z^* but of fixed Δ, where

$$z^* = 1 + (mB)^{-2/3}\Delta \tag{18}$$

and

$$P_m \sim \left(\int_{-\infty}^{z_-} + \int_{z_-}^{1}\right) J_{mB}[mBz + (mB)^{1/3}(-\Delta z)]\, dz$$

with z_- lying in the overlap region between the transition region $[\Delta = O(1)]$ and the "subsonic" region $[z = O(1)]$. In the second integral, put $u = (mB)^{2/3}(1 - z)$, use the assumed asymptotic behavior of $S(z)$ as $z \to 1$ and the leading order representation of $J_{mB}[mB - (mB)^{1/3}(u + \Delta)]$ in terms of an Airy function, and finally observe that since z_- lies in the overlap domain one must have $(1 - z_-) >> (mB)^{-2/3}$. Then the second integral gives

$$P_m \sim \frac{2^{1/3}\bar{S}}{(mB)^{(2\mu+3)/3}} \int_0^\infty u^\mu Ai[2^{1/3}(u + \Delta)\, du \tag{19}$$

For the first integral one proceeds as in the subsonic case and gets a contribution that, for z_- in the overlap domain, is exponentially small compared with the algebraically small expression, Eq. (19). Therefore we claim that, to leading order, Eq. (19) is the required description of the transition across the Mach condition. This is confirmed by asymptotic matching, as follows.

Let $\Delta \to +\infty$ and use the asymptotic form of $Ai(x)$ for $x \to +\infty$. This leads to an integral

$$I = \int_0^\infty \frac{x^\mu}{(x+1)^{1/4}} \exp\left[-\frac{(2\Delta)^{3/2}}{3}(x+1)^{3/2}\right] dx$$
$$\sim \int_0^\infty x^\mu \exp\left[-\frac{(2\Delta)^{3/2}}{3}\left(1 + \frac{3x}{2}\right)\right] dx$$
$$= \exp\left(-\frac{(2\Delta)^{3/2}}{3}\right) \frac{\mu!}{(2^{1/2}\Delta^{3/2})^{\mu+1}}$$

by the usual arguments, and hence Eq. (19) reduces, as $\Delta \to +\infty$, to

$$P_m \sim \frac{\bar{S}\mu!}{\pi^{1/2} 2^{(5/4)+(\mu/2)} (mB)^{(2\mu+3)/3} \Delta^{(\mu/2)+(3/4)}} \exp\left(-\frac{(2\Delta)^{3/2}}{3}\right) \tag{20}$$

On the other hand, as $z^* \to 1$ we have $\beta_t \sim (2\Delta)^{1/2}(mB)^{-1/3}$, and when we insert this in Eq. (16) and expand, we again get precisely Eq. (20). Thus Eqs. (16) and (19) match inboard of the Mach radius. Further, when $\Delta \to -\infty$ we have, from Eq. (19),

$$P_m \sim \frac{2^{1/3}\bar{S}}{(mB)^{(2\mu+3)/3}} |\Delta|^{\mu+1} \int_0^\infty x^\mu Ai[2^{1/3}|\Delta|(x-1)]\,dx$$
$$\sim \frac{2^{1/3}\bar{S}}{(mB)^{(2\mu+3)/3}} |\Delta|^{\mu+1} \int_{-\infty}^\infty Ai[2^{1/3}|\Delta|x]\,dx$$
$$= \frac{\bar{S}|\Delta|^\mu}{(mB)^{(2\mu+3)/3)}} \qquad (21)$$

where the arguments are those used in Sec. II earlier and in Sec. III. But setting $z^* = 1 + (mB)^{-2/3}\Delta$ in Eq. (17) and letting $(mB) \to \infty$ with Δ fixed gives again precisely Eq. (21), so that Eqs. (17) and (19) also match outboard of the Mach radius.

The function in Eq. (19) is tabulated only for the case $\mu = 0$, and Figs. 3a and 3b show plots derived from those tables for that case. The figures show a normalized SPL as a function of z^*, with curves corresponding to Eqs. (16), (17), and (19), for the cases $mB = 10$ and $mB = 100$. The subsonic approximation is singular like $\beta_t^{-3/2}$ as $\beta_t \to 0$ and greatly overpredicts the harmonic levels near the transition. On the other hand, the Mach-radius field, while remaining finite as $z^* \to 1$ (when $\mu = 0$), underestimates the field by a numerical factor. We have, for $\mu = 0$,

$$P_m \sim \frac{1}{mB} \int_{2^{1/3}\Delta}^\infty Ai(x)\,dx$$

and the overall maximum value of P_m occurs when $2^{1/3}\Delta = a_1 = -2.3381$, the first zero of $Ai(x)$. Then

$$(P_m)_{\max} \sim \frac{1}{mB} \int_{a_1}^\infty Ai(x)\,dx$$

and the integral has a value of about 1.27. For most practical purposes, this is acceptably close to 1, and so clearly the smoothing offered by Eq. (19) is much more necessary in relation to the subsonic approximation Eq. (16) than to the Mach-radius approximation Eq. (17).

For many purposes, it is convenient to have a composite approximation, which reproduces Eq. (16), (17), or (19) as appropriate and provides a smooth transition between them. Adding Eqs. (16), (17), and (19) and subtracting their common parts gives one such approximation, but it is slightly simpler to form composites of this kind for $z^* > 1$ and $z^* < 1$ separately; thus for $z^* > 1$ (subsonic radiation conditions)

$$(mB)P_m^{\text{comp}} = \int_{2^{1/3}\Delta}^\infty Ai(x)\,dx + \frac{\exp[mB(\tanh\beta_t - \beta_t)]}{(2\pi mB)^{1/2}\tanh^{3/2}\beta_t} \frac{-\exp[-(2\Delta)^{3/2}/3]}{\pi^{3/2}2^{5/4}\Delta^{3/4}} \qquad (22)$$

while for $z^* < 1$ (supersonic radiation conditions)

$$(mB)P_m^{\text{comp}} = \int_{2^{1/3}\Delta}^\infty Ai(x)\,dx + z^* - 1 \qquad (23)$$

where again we take $\mu = 0$ in Eq. (19). Equations (22) and (23) provide a continuous transition across $z^* = 1$, and although the transition is not smooth, the slight lack of smoothness is not evident in the logarithmic plots of Figs. 4a and 4b. In those figures, $mB = 10$ and 100, and the composite expressions are seen to provide a uniformly valid approximation of adequate accuracy.

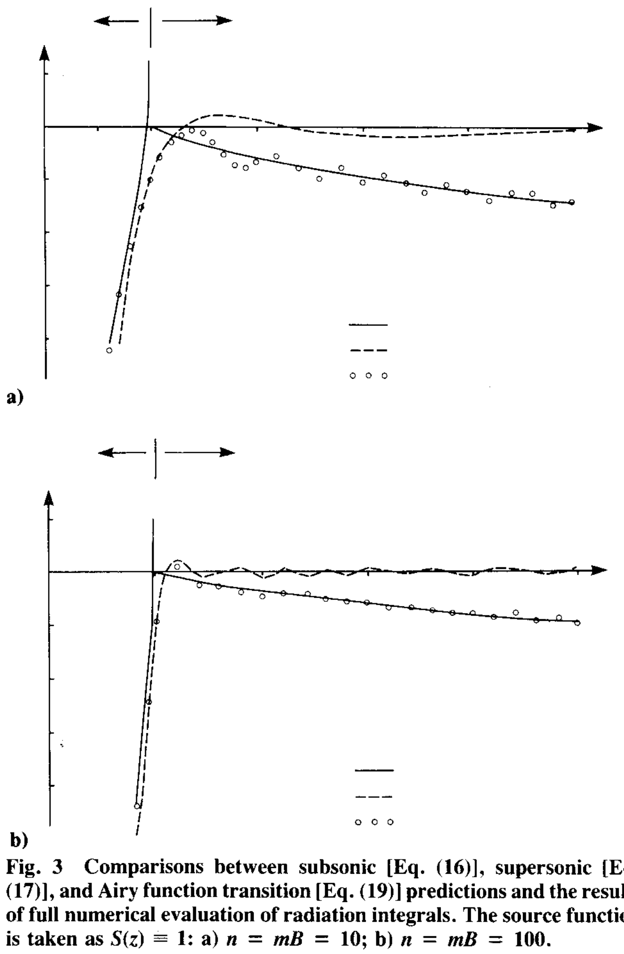

Fig. 3 Comparisons between subsonic [Eq. (16)], supersonic [Eq. (17)], and Airy function transition [Eq. (19)] predictions and the results of full numerical evaluation of radiation integrals. The source function is taken as $S(z) \equiv 1$: a) $n = mB = 10$; b) $n = mB = 100$.

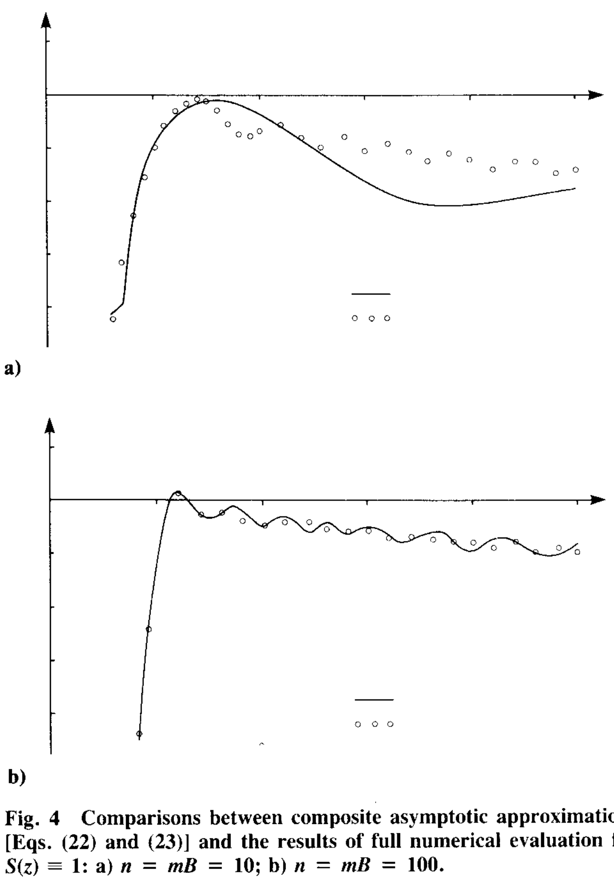

Fig. 4 Comparisons between composite asymptotic approximations [Eqs. (22) and (23)] and the results of full numerical evaluation for $S(z) \equiv 1$: a) $n = mB = 10$; b) $n = mB = 100$.

Generally the loading and thickness distributions will taper to zero at the tip, so that $\mu > 0$ will be more appropriate for practical cases, and then the function of Δ defined by Eq. (19) must be calculated by numerical means for each appropriate μ. There is no alternative if one wants an analytical description of the transition; the dominant radiation from the Mach radius is necessarily described by the Airy function, and the tip distribution of the source strength by some assumed power law $S(z) \sim \bar{S}(1 - z)^{\mu}$, and it is inevitable that the transition of the Mach radius across the tip will be described by nothing simpler than the Airy function moment (perhaps of fractional order μ) in Eq. (19). Calculation of this moment is actually not much simpler than calculation of the original integral, although there is some simplification in the fact that the integral in Eq. (19) depends on one variable only, whereas that in Eq. (1) depends on three variables.

IV. Conclusions

The asymptotic theory of propeller noise for the large-blade-number limit $B \to \infty$ has been extended here to include two significant higher- order effects. First, for supersonic radiation conditions we have shown that the first correction to the dominant Mach radius mechanism comes from the blade tip, and the explicit formulas found here give highly accurate predictions when compared with numerical evaluation of the radiation integrals and leave little doubt that the theory reproduces all significant effects that are present in the full integrals. Moreover, of course, the *locations* of the radiation sources for each operating condition and each observer direction have been identified, locations that are not revealed by computation alone. Second, we have provided expressions for the continuous transition through the Mach condition $z^* = 1$, at which Mach radius and tip coincide, where the subsonic formulas are badly inaccurate, and have shown that the transition expressions give the necessary smoothing required for practical use. We emphasize again that all spanwise interference effects (arising from blade sweep and from chordwise non-compactness) have been ignored and will be dealt with elsewhere.

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